Name of College - S.S.College, J.Bad
Deptt - Mathematics
Topic - Problem on Indeterminate Form (contd)
Differential Calculus
Class - B.Sc I (Hons)

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Time - 10.15 A.M To 11 A.M
11.00 A.M To 11.45 A.M

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By - Samrendra Kumar

Imp
Problem :-> Evaluate $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

Let $y = \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

Taking log of both sides

$$\log y = \lim_{x \to 0} \log\left(\frac{\tan x}{x}\right)^{1/x}$$

[using $\log \frac{m}{n} = \log m - \log n$, $\log m^n = n \log m$.]

$$= \lim_{x \to 0} \frac{1}{x}\left[\log \tan x - \log x\right]$$

$$= \lim_{x \to 0} \frac{\log \tan x - \log x}{x} \quad \left[\frac{0}{0}\right]$$

Applying L-Hospital Rules.

$$\log y = \lim_{x \to 0} \frac{\frac{1}{\tan x}\sec^2 x - \frac{1}{x}}{1}$$

$$= \lim_{x \to 0} \frac{1}{\sin x \cos x} - \frac{1}{x}$$

$$= \lim_{x \to 0} \frac{2}{\sin 2x} - \frac{1}{x}$$

$$= \lim_{x \to 0} \frac{2x - \sin 2x}{x \sin 2x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{2 - 2\cos 2x}{\sin 2x + 2x \cos 2x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{4 \sin 2x}{2\cos 2x + 2\left[\cos 2x - 2x \sin 2x\right]}$$

$$= \lim_{x \to 0} \frac{4 \sin 2x}{2\cos 2x + 2\cos 2x - 4x \sin 2x}$$

$$= 0$$

$\Rightarrow \log y = 0$
$\Rightarrow e^0 = y \quad \Rightarrow y = 1$

Problem - Evaluate $\lim_{x\to 0}\left[\frac{\tan x}{x}\right]^{1/x^2}$

Let $y = \lim_{x\to 0}\left[\frac{\tan x}{x}\right]^{1/x^2}$ Form $1^{\infty}$

Taking log of both sides,

$$\log y = \lim_{x\to 0}\frac{\log\tan x - \log x}{x^2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{2x}$$

$$= \lim_{x\to 0}\frac{\frac{1}{\sin x\cos x} - \frac{1}{x}}{2x}$$

$$= \lim_{x\to 0}\frac{\frac{2}{\sin 2x} - \frac{1}{x}}{2x}$$

$$= \lim_{x\to 0}\frac{2x - \sin 2x}{2x^2\sin 2x} \quad \left\{\frac{0}{0}\right]$$

Again Applying L-Hospital Rule.

$$= \lim_{x\to 0}\frac{2 - 2\cos 2x}{2\left[2x\sin 2x + 2x^2\cos 2x\right]}$$

$$= \lim_{x\to 0}\frac{1-\cos 2x}{2x\sin 2x + 2x^2\cos 2x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{2\sin 2x}{2\left[\sin 2x + 2x\cos 2x\right] + 2\left[2x\cos 2x - 2x^2\sin 2x\right]}$$

$$\Rightarrow \log y = \lim_{x\to 0} \frac{2\sin 2x}{2\sin 2x + 4x\cos 2x + 4x\cos 2x - 4x^2\sin 2x}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{\sin 2x + 4x\cos 2x - 2x^2\sin 2x}$$

$$= \lim_{x\to 0} \frac{2\cos 2x}{2\cos 2x + 4\cos 2x - 8x\sin 2x - 4x\sin 2x - 4x^2\cos 2x}$$

$$= \frac{2}{2+4-0-0-0}$$

$$= \frac{2}{6} = \frac{1}{3}$$

$$\therefore \log y = \frac{1}{3} \Rightarrow y = e^{1/3}$$

Evaluate

$$\lim_{x\to 0} \log_{\tan x} \tan 2x \qquad \log_a b = \frac{\log b}{\log a}$$

Solution :→ $\lim_{x\to 0} \log_{\tan x} \tan 2x$

$$= \lim_{x\to 0} \frac{\log \tan 2x}{\log \tan x} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{\tan 2x} \cdot \sec^2 2x \cdot 2}{\frac{1}{\tan x} \cdot \sec^2 x}$$

$\sin x \cos x$

$= \lim_{x\to 0} \frac{2\sin x\cos x}{\sin 2x \cos 2x}$

$= 2\lim_{x\to 0} \frac{\sin 2x}{\sin 4x} \quad [0/0]$

$= 2\lim_{x\to 0} \frac{2\cos 2x}{4\cos 4x}$

$= \lim_{x\to 0} \frac{4\cos 2x}{4\cos 4x} = 1$

Evaluate $\lim_{x\to 0} \log_{\sin x} \sin 2x$

Solution :- $\lim_{x\to 0} \log_{\sin x} \sin 2x$

$= \lim_{x\to 0} \frac{\log \sin 2x}{\log \sin x} \quad \left[\frac{\infty}{\infty}\right]$

$= \lim_{x\to 0} \frac{\frac{1}{\sin 2x} \cdot 2\cos 2x}{\frac{1}{\sin x} \cdot \cos x}$ | Applying L-Hospital Rule

$= \lim_{x\to 0} \frac{2\cos 2x \times \sin x}{\sin 2x \cos x}$

$= \lim_{x\to 0} \frac{2\cos 2x \cdot \sin x}{2\sin x\cos x \cdot \cos x} = \frac{2\times 1}{2\times 1}$

Evaluate $\lim_{x\to 0} \log_{\tan^2 x} (\tan^2 2x)$

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Solution $\rightarrow \lim_{x \to 0} \log_{\tan x} \tan^2 x$

[using $\log_a^b = \frac{\log_e^b}{\log_e^a}$]

$$= \lim_{x \to 0} \frac{\log \tan^2 2x}{\log \tan^2 x}$$

using $\log m^n = n \log$

$$= \lim_{x \to 0} \frac{2 \log \tan 2x}{2 \log \tan x}$$

$$= \lim_{x \to 0} \frac{\log \tan 2x}{\log \tan x} \quad \left[\text{Form } \frac{\infty}{\infty}\right]$$

using L-Hospital Rule

$$= \lim_{x \to 0} \frac{\frac{\sec^2 2x \times 2}{\tan 2x}}{\frac{1}{\tan x} \sec^2 x}$$

$$= 2 \lim_{x \to 0} \frac{\sin x \cos x}{\sin 2x \cos 2x}$$

$$= 2 \lim_{x \to 0} \frac{\sin 2x}{\sin 4x} \quad [0/0]$$

$$= 2 \lim_{x \to 0} \frac{2 \cos 2x}{4 \cos 4x}$$

using L-Hospital Rule.

$=$

Evaluate

$$\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x}$$

Since $a^x = e^{x\log a}$

Also $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

<u>Solution</u> :→ we have to find

$$\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x}$$

for $(1+x)^{1/x} - e$

$$= e^{\frac{1}{x}\log(1+x)} - e$$

$$= e^{\frac{x - x^2/2 + x^3/3 \cdots}{x}} - e$$

$$= e^{1 - x/2 + x^2/3 \cdots} - e$$

$$= e \cdot e^{-[x/2 - x^2/3 \cdots]} - e$$

$$= e\left[e^{-(x/2 - x^2/3 + x^3/4 \cdots} - 1\right] - e$$

Now we know $e^x = 1 + x + x^2/\underline{|2} + x^3/\underline{|3} \cdots$

$$e^{-x} = 1 - x + x^2/\underline{|2} - \frac{x^3}{\underline{|3}} + \cdots$$

$$= e\left\{1 + \left(-x/2 + x^2/3 - x^3/4 + \cdots\right) + \frac{1}{\underline{|2}}\left(-x/2 + x^2/3 - \frac{x^3}{4} + \cdots\right.\right.$$
$$\left. + \frac{1}{\underline{|3}}\left(-x/2 + x^2/3 - x^3/4 + \cdots\right)^3 \cdots\right\} - e$$

$$= ex\left[\left(-\frac{1}{2}+\frac{x}{3}-\cdots\right)+\frac{x}{\lfloor 2}\left(-\frac{1}{2}+\cdots\right)^2\right]$$

$$\therefore \lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$$

$$= \lim_{x\to 0}\frac{\cancel{e}\cancel{x}\left[\left\{-\frac{1}{2}+\frac{x}{3}\cdots\right)+\frac{x}{\lfloor 2}\left(-\frac{1}{2}+\cdots\right)^2\right]}{\cancel{x}}$$

$$= e\left[\left(-\frac{1}{2}+0\cdots\right)+0\cdots\right]$$

$$= -\frac{e}{2}$$

V. Imp.

Show that.

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{ex}{2}}{x^2} = \frac{11e}{24}$$

<u>Solution</u> :→ $\lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{ex}{2}}{x^2}$

$$= \lim_{x\to 0}\frac{e^{\frac{\log(1+x)}{x}}-e+\frac{ex}{2}}{x^2}$$

$a^x = e^{x\log a}$

$$= \lim_{x\to 0} \frac{e^{\frac{1}{x}\left(x - x^2/2 + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)} - e + \frac{1}{2}ex}{x^2}$$

$$= \lim_{x\to 0} \frac{e^{\left(1 - x/2 + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right)} - e + \frac{ex}{2}}{x^2}$$

$$= \lim_{x\to 0} \frac{e \cdot e^{\left(-x/2 + x^2/3 - \frac{x^3}{4} + \cdots\right)} - e + \frac{ex}{2}}{x^2}$$

$$= e \lim_{x\to 0} \frac{e^{\left(-x/2 + x^2/3 - \frac{x^3}{4} \cdots\right)} - 1 + \frac{x}{2}}{x^2}$$

$$= e \lim_{x\to 0} \frac{\left\{1 + \left(-x/2 + x^2/3 - \frac{x^3}{4} \cdots\right) + \frac{1}{2!}\left(-x/2 + \frac{x^2}{3} - \frac{x^3}{4} +\right)^2 + \frac{1}{3!}\left(-x/2 + x^2/3 - \frac{x^3}{4} + \cdots\right)^3\right\} - 1 + \frac{1}{2}x}{x^2}$$

$$= e \lim_{x\to 0} \frac{\left(-x/2 + x^2/3 - \frac{x^3}{4} + \cdots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right)^2 + \frac{1}{6}\left(-x/2 + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right)^3 + \frac{1}{2}x}{x^2}$$

$$= e \lim_{x\to 0} \frac{x\left[\left(-1/2 + \frac{x}{3} - \frac{x^2}{4} + \cdots\right) + \frac{1}{2}x^2\left(-1/2 + \frac{x}{3} - \frac{x^2}{4} + \cdots\right)^2 + \frac{x^3}{6}\left(-1/2 + x/3 - \frac{x^2}{4} \cdots\right)^3 + \frac{1}{2}\right]}{x^2}$$

$$= e^{\lim_{x\to 0} \frac{\left(\frac{x}{3}-\frac{x^2}{4}+\cdots\right)+\frac{1}{2}x\left(-\frac{1}{2}+\frac{x}{3}-\frac{x^2}{4}+\cdots\right)^2 + \frac{x^3}{6}\left(-\frac{1}{2}+\frac{x}{3}-\frac{x^2}{4}+\cdots\right)}{x}}$$

$$= e^{\left(\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{2}\right)}$$

$$= e^{\frac{8+3}{24}}$$

$$= e^{\frac{11}{24}} = \frac{11}{24}e.$$